

A STATE POLYTOPE DECOMPOSITION FORMULA

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ABSTRACT. We give a decomposition formula for computing the state polytope of a reducible variety in terms of the state polytopes of its components.

1. INTRODUCTION

The state polytope of an ideal encodes much information about the scheme it defines. Let V be a vector space over an algebraically closed field k of characteristic zero. Given a rational representation W of $SL(V)$ and a maximal torus $T \subset SL(V)$, Kempf [Kem78, § 3] defined the *state* of $w \in W$ (with respect to T) to be the set of the characters $\chi \in X(T)$ such that $w_\chi \neq 0$ where w_χ is the projection of w in the weight space W_χ . Given a projective variety $X \subset \mathbb{P}(V)$ and a choice of homogeneous coordinates, Bayer and Morrison in [BM88] defined the *m th state polytope* of $X \subset \mathbb{P}(V)$ to be the convex hull of the states of (any affine point over) the m th Hilbert point $[X]_m \in \mathbb{P}\left(\bigwedge^{P(m)} S^m V^*\right)$, where $P(t) \in \mathbb{Q}[t]$ is the Hilbert polynomial of X and m is a positive integer bigger than or equal to the Castelnuovo-Mumford regularity of X . The state polytope of a homogeneous ideal $I \subset k[x_0, \dots, x_n]$ is the state polytope of the projective variety it defines. The relation between the state polytopes and the Gröbner theory is described by:

Theorem. [BM88, Theorem 3.1] *There is a natural one-to-one correspondence between the initial ideals and the vertices of the state polytope.*

In fact, for the purpose of this article, it is convenient to take the the following definition of state polytope [Stu96].

Definition 1.1. Given an ideal $I \subset S = k[x_0, \dots, x_n]$ and $m \geq \text{reg}(I)$, the *m th state polytope* is defined and denoted by

$$(1) \quad \text{State}_m(I) := \text{Conv} \left\{ \sum_{x^\alpha \in \text{in}_{\prec}(I)_m} \alpha \mid \prec \text{ is a monomial order} \right\}.$$

By considering the weight space decomposition of $\bigwedge^{P(m)} S^m V^*$, one can associate the characters in the state of a Hilbert point $[X]_m$ and the corresponding points in the state polytope $\text{State}_m(I_X)$: see [MS11, Section 3.1]. In particular, the trivial character corresponds to the *barycenter* of which coordinates are all $\frac{mQ(m)}{\dim V}$ where $Q(m) = \dim_k(I_X)_m$. Apparent from the definition is that the state polytope can be computed from the universal Gröbner basis. More importantly, $\text{State}_m(I_X)$

Date: April 2, 2013.

The first author was supported by the following grants funded by the government of Korea: NRF grant 2011-0030044 (SRC-GAIA), NRF grant 2011-0005072 and the Korea Institute for Advanced Study (KIAS) grant.

determines the semistability of the Hilbert point $[X]_m$ with respect to the chosen basis. This is a direct consequence of the Mumford's numerical criterion [Mum65]: $[X]_m$ is T -semistable (resp. T -stable) if and only if the state polytope in $X(T)$ (resp. interior of the polytope) contains the trivial character. This condition is equivalent to the state polytope (1) (resp. its interior) containing the barycenter. The upshot is that, by computing the universal Gröbner basis (with a computer algebra system if and when convenient), one can determine the semistability with respect to the given coordinates. Morrison and Swinarski worked out several interesting examples in [MS11] which are important in view of the log minimal model program for \overline{M}_g (Hassett-Keel program).

In this paper, we shall consider how one can more efficiently compute the state polytope of certain reducible varieties. More precisely, we give a formula for the state polytope of a variety in terms of the state polytopes of its subvarieties.

Theorem 1.2. *Let $X = \cup_{i=1}^{\ell} X_i \subset \mathbb{P}^n$ be a closed subvariety defined by a homogeneous ideal $I_X = \cap_i I_{X_i}$. Suppose that there is a homogeneous coordinate system*

$$x_0, \dots, x_{n_1}, x_{n_1+1}, \dots, x_{n_2}, x_{n_2+1}, \dots, x_{n_\ell} = x_n$$

such that

$$X_i \subset \{x_0 = \dots = x_{n_{i-1}-1} = 0, x_{n_i+1} = x_{n_i+2} = \dots = x_n = 0\}$$

where $n_{-1} = 0$ and $n_\ell = n$. Then the state polytope of X is given by the following decomposition formula

(2)

$$\text{State}_m(I_X) = \sum_{i=1}^{\ell} \text{State}_m(I_{X_i} \cap k[x_{n_{i-1}}, \dots, x_{n_i}]) + \sum_{i=1}^{\ell-1} \text{State}_m(T_i \cap k[x_{n_{i-1}}, \dots, x_n])$$

where $T_i = \langle x_{n_{i-1}}, \dots, x_{n_i-1} \rangle \langle x_{n_i+1}, \dots, x_n \rangle$. Here, $\text{State}_m(I_{X_i} \cap k[x_{n_{i-1}}, \dots, x_{n_i}])$ is regarded as a convex polytope in the subspace

$$\{x_0 = \dots = x_{n_{i-1}-1} = 0, x_{n_i+1} = x_{n_i+2} = \dots = x_n = 0\} \subset \mathbb{R}^{n+1}.$$

Similarly, $\text{State}_m(T_i \cap k[x_{n_{i-1}}, \dots, x_n])$ is also regarded as a convex polytope in the relevant vector subspace.

We shall give a proof of Theorem 1.2 in § 3.

Remark 1.3. Note that the second term of (2) is zero dimensional since T_i are monomial ideals. We shall reserve the letter γ to denote it.

In fact, the polytope decomposition in Theorem 1.2 is sharp in the following sense:

Corollary 1.4. *Retain notations from Theorem 1.2. Let \mathcal{V}_i denote the set of vertices $\text{State}_m(I_{X_i} \cap k[x_{n_{i-1}}, \dots, x_{n_i}])$, $i = 1, \dots, \ell$. Then the vertices of $\text{State}_m(I_X)$ are precisely*

$$\left\{ \gamma + \sum_{i=1}^{\ell} v_i \mid v_i \in \mathcal{V}_i \right\}.$$

Proof. We shall prove the case $\ell = 2$, whence the general cases follow inductively. Let $\{v_i\}_{i=1}^r$ be vertices of $\text{State}_m(I_{X_1} \cap k[x_0, \dots, x_{n_1}])$ and $\{w_i\}_{i=1}^s$, those of $\text{State}_m(I_{X_2} \cap k[x_{n_1}, \dots, x_{n_2}])$. If $v_1 + w_1$ is not a vertex of the Minkowski sum

of the state polytopes, then there exist λ_{ij} for $i = 1, \dots, r, j = 1, \dots, s$ such that $\sum_{i,j} \lambda_{ij} = 1$, $0 \leq \lambda_{ij} \leq 1$, $\lambda_{11} = 0$ and

$$v_1 + w_1 = \sum_{i,j} \lambda_{ij} (v_i + w_j).$$

By rearranging the terms, we have

$$(1 - \sum_j \lambda_{1j})v_1 - \sum_{i \neq 1} \lambda_{ij}v_i = \sum_{j \neq 1} \lambda_{ij}w_j + (\sum_i \lambda_{i1} - 1)w_1$$

which implies that x_{n_1} is the only nonzero coordinate of each side. Moreover, $1 - \sum_j \lambda_{1j} \neq 0$ or $\sum_i \lambda_{i1} - 1 \neq 0$ since $\sum_{i,j} \lambda_{ij} = 1$. Suppose the first and let $v_i = (v_{i0}, v_{i1}, \dots, v_{in_1})$. Then we have

$$v_{1k} = \sum_{i \neq 1} \mu_{ij}v_{ik} \quad \text{except for } k = n_1$$

where $\mu_{ij} = \lambda_{ij}/(1 - \sum_j \lambda_{1j})$ and $\sum_{i \neq 1} \mu_{ij} = 1$. Let $Q_1(m)$ be the dimension of $(I_{X_1} \cap k[x_0, \dots, x_{n_1}])_m$. Then the n_1 th coordinate also satisfies the above condition because

$$\begin{aligned} v_{1n_1} &= mQ_1(m) - \sum_{k=0}^{n_1-1} v_{1k} = mQ_1(m) - \sum_{k=0}^{n_1-1} \sum_{i \neq 1} \mu_{ij}v_{ik} \\ &= \sum_{i \neq 1} \mu_{ij}mQ_1(m) - \sum_{i \neq 1} \sum_{k=0}^{n_1-1} \mu_{ij}v_{ik} \\ &= \sum_{i \neq 1} \{\mu_{ij}(mQ_1(m) - \sum_{k=0}^{n_1-1} v_{ik})\} = \sum_{i \neq 1} \mu_{ij}v_{in_1} \end{aligned}$$

which means that

$$v_1 = \sum_{i \neq 1} \mu_{ij}v_i.$$

But this is a contradiction to the fact that v_1 is a vertex of $\text{State}_m(I_{X_1} \cap k[x_0, \dots, x_{n_1}])$. \square

We give two toy examples at the far ends of the spectrum, namely, monomial ideals and hypersurfaces (plane curves).

Example 1.5 (Monomial ideals). Let $X = X_1 \cup X_2 \cup X_3 \subset \mathbb{P}^3$ be a chain of \mathbb{P}^1 's:

$$X_1 = \{x_2 = x_3 = 0\}, \quad X_2 = \{x_0 = x_3 = 0\}, \quad X_3 = \{x_0 = x_1 = 0\}.$$

Then $I_{X_1} \cap k[x_0, x_1] = 0$ and it does not contribute to the state polytope. The other components do not contribute for the same reason, but the mixed terms

$$\{x_0x_2, x_0x_3\} \cup \{x_0x_3, x_1x_3\}$$

are precisely the monomial generators of the ideal of X , and the formula (2) holds.

More generally, if $X = \cup X_i$ as in Theorem 1.2 and each X_i are defined by monomials $M_{i\alpha}$,

$$\text{State}_m(I_{X_i} \cap k[x_{n_{i-1}}, \dots, x_{n_i}]) = \left\{ \sum_{\alpha} \log_X M_{i\alpha} \right\}$$

where $\log_x x^\alpha = \alpha$. The formula (2) implies that

$$\text{State}_m(I_X) = \left\{ \sum_i \sum_\alpha \log_x M_{i\alpha} + \sum_{x^\beta \in \sum_i T_i} \beta \right\}.$$

This is the same as

$$\text{State}_m(I_X) = \left\{ \sum_{x^\alpha \in (I_X)_m} \alpha \right\}$$

since

$$I_X = \cap_i I_{X_i} = \cap_i \langle \{M_{i\alpha}\}_\alpha, x_0, \dots, x_{n_{i-1}-1}, x_{n_i+1}, \dots, x_n \rangle = \langle \{M_{i\alpha}\}_{i,\alpha} \rangle + \sum_i T_i.$$

Note that the inclusion $M_{i\alpha} \in \cap_i I_{X_i}$ follows from the assumption that $M_{i\alpha}$ is not a power of x_{n_i} or of $x_{n_{i-1}}$.

Example 1.6 (Plane curves). We consider a simple example of two plane curves $E_1 = \{b^2c = a(a-c)(a-2c), d = e = 0\}$ and $E_2 = \{d^2c = e^2(e+1), a = b = 0\}$ meeting in one node. Let C denote the union of E_1 and E_2 . The 3rd state polytope of E_1 has three vertices

$$\{(3, 0, 0, 0, 0), (1, 0, 2, 0, 0), (0, 2, 1, 0, 0)\}$$

and that of E_2 has two vertices

$$\{(0, 0, 1, 2, 0), (0, 0, 0, 0, 3)\}.$$

Hence by the decomposition formula (Corollary 1.4), the 3rd state polytope of C has vertices

$$\{(3, 0, 1, 2, 0), (3, 0, 0, 0, 3), (1, 0, 3, 2, 0), (1, 0, 2, 0, 3), (0, 2, 2, 2, 0), (0, 2, 1, 0, 3)\}$$

translated by $\gamma = (11, 11, 4, 11, 11)$, the sum of the exponent vectors of

$$T = \{a^2d, abd, acd, ad^2, ade, a^2e, abe, ace, ae^2, b^2d, bcd, bd^2, bde, b^2e, bce, be^2\}.$$

This agrees with the direct computation of the 3rd state polytope of the ideal of C :

$$\langle be, ae, bd, ad, -cd^2 + e^3 + e^2, a^3 - 3a^2c - b^2c + 2ac^2 \rangle$$

Here we demonstrate the output of the Macaulay 2 package `StatePolytope` written by D. Swinarski.

```
i2 : R = QQ[a,b,c,d,e];
i3 : I1 = ideal(b^2*c - a*(a-c)*(a-2*c))
          3      2      2      2
o3 = ideal(- a  + 3a c + b c - 2a*c )
o3 : Ideal of R
i4 : I2 = ideal(d^2*c-e^2*(e+1))
          2      3      2
o4 = ideal(c*d  - e  - e )
o4 : Ideal of R
i5 : I = intersect(I1+ideal(d,e),I2+ideal(a,b))
          2      3      2      3      2      2      2
o5 = ideal (b*e, a*e, b*d, a*d, - c*d  + e  + e , a  - 3a c - b c + 2a*c )
o5 : Ideal of R
i6 : statePolytope(3,I)
LP algorithm being used: "cddgmp".
polymake: used package cddlib
Implementation of the double description method of Motzkin et al.
Copyright by Komei Fukuda.
```

\protect\vrule width0pt\protect\href{http://www.ifor.math.ethz.ch/~string~fukuda/cdd_home/cdd.html}{http://www.ifor.math.ethz.ch/~string~fukuda/cdd_home/cdd.html}

VERTICES

1 14 11 5 13 11
 1 14 11 4 11 14
 1 12 11 6 11 14
 1 11 13 5 11 14
 1 12 11 7 13 11
 1 11 13 6 13 11

o6 = {{14, 11, 5, 13, 11}, {14, 11, 4, 11, 14}, {12, 11, 6, 11, 14},

 {11, 13, 5, 11, 14}, {12, 11, 7, 13, 11}, {11, 13, 6, 13, 11}}

Example 1.7. Let R be a rational curve with a rhamphoid cusp. It is of arithmetic genus two and admits a \mathbb{G}_m action with two fixed points one of which is the cusp. The action comes from the automorphism of its normalization. Let C be a genus two curve obtained by attaching two copies R_1, R_2 of R at the smooth fixed points, say $p_i \in R_i$, $i = 1, 2$. We bicanonically embed C in \mathbb{P}^8 and consider its state polytope. Note that R_1 can be parametrized by

$$\begin{array}{ccc} \mathbb{P}^1 & \rightarrow & \mathbb{P}^8 \\ [s, t] & \mapsto & [s^6, s^4t^2, s^2t^4, st^5, t^6, \mathbf{0}_4] \end{array}$$

which has rhamphoid cusp at $[1, 0, \dots, 0]$ and $[\mathbf{0}_4, 1, \mathbf{0}_4]$ is fixed under the automorphism where $\mathbf{0}_i = (\underbrace{0, \dots, 0}_i)$. R_2 is parametrized similarly, from which we can

compute its defining saturated ideal. The 6th state polytope of $I_{R_1} \cap k[x_0, \dots, x_4]$ has 51 vertices

$\{(216, 191, 206, 206, 231, \mathbf{0}_4), (216, 191, 191, 236, 216, \mathbf{0}_4), \dots, (181, 248, 210, 180, 231, \mathbf{0}_4)\}$

The state polytope of $I_{R_2} \cap k[x_4, \dots, x_8]$ is the mirror flip of the above with respect to x_4 . By the decomposition formula, the 6th state polytope P of C is $\gamma = (1750, 1750, 1750, 1750, 1504, 1750, 1750, 1750, 1750)$ -translate of the convex hull of $51 \cdot 51 = 2601$ vertices obtained by choosing one from each state polytope and adding them up, and the barycenter of this state polytope is $v = (1956, 1956, \dots, 1956)$. Since we can split $v - \gamma$ into the two symmetric points $v_1 = (206, 206, 206, 206, 226, \mathbf{0}_4)$ and $v_2 = (\mathbf{0}_4, 226, 206, 206, 206, 206)$, we just have to check that v_1 is contained in 6th state polytope of R_1 to show that v is in P . But this can be easily checked by using the function `contains` of the polyhedra package in Macaulay 2 as follows.

```
i3 : R=QQ[a,b,c,d,e];
i4 : Q=QQ[s,t];
i5 : f=map(Q,R,{s^6,s^4*t^2,s^2*t^4,s*t^5,t^6});
o5 : RingMap Q <--- R
i6 : I=ker f;
o6 : Ideal of R
i7 : L=statePolytope(6,I);
i8 : P1=convexHull transpose matrix L;
i9 : v1=transpose matrix {{206,206,206,206,226}};
      5      1
o9 : Matrix ZZ <--- ZZ
i10 : contains(P1,v1)
o10 = true
```

Hence, we conclude that \mathbf{v} is contained in \mathbf{P} . On the other hand, the ideal of \mathbf{C} is simple enough so that its state polytope can be directly computed by using the state polytope package of Macaulay 2, which agrees with the result obtained above.

The assumption on the existence of the coordinate system as in Theorem 1.2 may seem quite restrictive, but we shall see that it is satisfied by several important classes of varieties including, most notably, the pluricanonical images of the generic members of the boundary of $\overline{\mathcal{M}}_g$. We shall give a few interesting examples in this vein in § 4.

Hilbert-Mumford index can also be computed by a similar decomposition formula:

Proposition 1.8. *Let X be as in Theorem 1.2 and $\rho : \mathbb{G}_m \rightarrow \mathrm{GL}_{n+1}$ be a 1-parameter subgroup of GL_{n+1} diagonalized by $\{x_0, \dots, x_n\}$ with weights (r_0, \dots, r_n) and ρ_i be the restriction of ρ to $\mathrm{GL}(kx_{n_{i-1}} + \dots + kx_{n_i})$. Then the Hilbert-Mumford index $\mu([C]_m, \rho)$ of the m th Hilbert point of X with respect to ρ is given by*

$$\mu([X]_m, \rho) = \sum_{i=1}^l \mu([X_i]_m, \rho_i) - \sum_{i=1}^l \left(\frac{mP_i(m)}{n_i - n_{i-1} + 1} \sum_{k=n_{i-1}}^{n_i} r_k \right) + \frac{mP(m)}{n+1} \sum_{i=0}^n r_i + m \sum_{i=1}^{\ell-1} r_{n_i}$$

where $P(m)$ is the Hilbert polynomial of $I_X \subset k[x_0, \dots, x_n]$ and $P_i(m)$, the Hilbert polynomial of $I_{X_i} \subset k[x_{n_{i-1}}, \dots, x_{n_i}]$ regarded as an ideal in $k[x_{n_{i-1}}, \dots, x_{n_i}]$.

We shall prove this proposition in § 3.

2. DECOMPOSITION FORMULA FOR INITIAL IDEALS AND THE HILBERT-MUMFORD INDEX

First we shall prove a key lemma on initial ideals from which the main theorem follows with some simple observations regarding the monomial orders.

Let $X = Y \cup Z$ be a closed subvariety in \mathbb{P}^n and suppose that there exists a homogeneous coordinate system x_0, x_1, \dots, x_n of \mathbb{P}^n such that

$$\begin{aligned} Y &\subset L_1 := \{x_0 = \dots = x_{l-1} = 0\} \cong \mathbb{P}^{n-l} \\ Z &\subset L_2 := \{x_{l+1} = \dots = x_n = 0\} \cong \mathbb{P}^l. \end{aligned}$$

In particular, $Y \cap Z = \{p\}$ where p is the unique point in $L_1 \cap L_2$ whose coordinates are all zero except x_l . Let I_X be the ideal of X in $k[x_0, x_1, \dots, x_n]$ and I_Y and I_Z be the ideals of Y and Z respectively.

Lemma 2.1. *For a fixed monomial order \prec on $k[x_0, \dots, x_n]$, the initial ideal of I is given by*

$$\mathrm{in}_{\prec}(I_X) = \langle \mathrm{in}_{\prec}(I_Y \cap k[x_l, \dots, x_n]) \rangle + \langle \mathrm{in}_{\prec}(I_Z \cap k[x_0, \dots, x_l]) \rangle + T$$

where $T = \langle x_0, \dots, x_{l-1} \rangle \langle x_{l+1}, \dots, x_n \rangle$. Note that $\mathrm{in}_{\prec}(I_Y \cap k[x_l, \dots, x_n])$ is computed as an ideal of $k[x_l, \dots, x_n]$ and $\langle \mathrm{in}_{\prec}(I_Y \cap k[x_l, \dots, x_n]) \rangle$ is the ideal in $k[x_0, \dots, x_n]$ it generates.

Proof. We first note that T is contained in $\mathrm{in}_{\prec}(I_X)$ since it is a monomial ideal contained in I_X . Let $x^\alpha = \prod x_i^{\alpha_i}$ be a monomial in $\mathrm{in}_{\prec}(I_X)_m$ i.e. $x^\alpha = \mathrm{in}_{\prec}(f)$ for some $f \in I_X$. If $x^\alpha \notin T$, then x^α is contained in $k[x_0, \dots, x_l]$ or $k[x_l, \dots, x_n]$. If $x^\alpha \in k[x_0, \dots, x_l]$, then $\mathrm{in}_{\prec}(g) = x^\alpha$ where $g(x_0, \dots, x_l) = f(x_0, \dots, x_l, 0, \dots, 0)$. But $g \in I_Z \cap k[x_0, \dots, x_l]$, so x^α is contained in $\mathrm{in}_{\prec}(I_Z \cap k[x_0, \dots, x_l])_m$. Similarly,

if $x^\alpha \in k[x_1, \dots, x_n]$, then $x^\alpha \in \text{in}_{\prec}(I_Y \cap k[x_1, \dots, x_n])_m$. This proves that the left hand side is contained in the right.

To see the other inclusion, suppose that $x^\alpha \in \text{in}_{\prec}(I_Y \cap k[x_1, \dots, x_n])_m$ i.e., $x^\alpha = \text{in}_{\prec}(f)$ for some $f \in I_Y \cap k[x_1, \dots, x_n]$. Since f vanishes at $p \in Y$, it cannot have the term x_l^i , so each term of f has the variable x_i for some $i > l$. This implies that f vanishes on Z so that $f \in I_X$ and $x^\alpha \in \text{in}_{\prec}(I_X)$. Similarly, $\text{in}_{\prec}(I_Z \cap k[x_0, \dots, x_l])_m \subset \text{in}_{\prec}(I_X)_m$ and this completes the proof. \square

We obtain the following corollary by induction.

Corollary 2.2. *Retain the notations from the main Theorem 1.2. For any monomial order \prec on $k[x_0, \dots, x_n]$, we have*

$$\begin{aligned} \text{in}_{\prec}(I_X) &= \sum_{i=1}^{\ell} \langle \text{in}(I_{X_i} \cap k[x_{n_{i-1}}, \dots, x_n]) \rangle + \sum_{i=1}^{l-1} T_i \\ &= \sum_{i=1}^{\ell} \langle \text{in}(I_X \cap k[x_{n_{i-1}}, \dots, x_n]) \rangle + \sum_{i=1}^{l-1} T_i. \end{aligned}$$

Lemma 2.3. *Let $\Sigma_{Y,m}$ and $\Sigma_{Z,m}$ be defined*

$$\begin{aligned} \Sigma_{Y,m} &= k[x_1, \dots, x_n]_m \setminus \text{in}_{\prec}(I_Y \cap k[x_1, \dots, x_n])_m \\ \Sigma_{Z,m} &= k[x_0, \dots, x_l]_m \setminus \text{in}_{\prec}(I_Z \cap k[x_0, \dots, x_l])_m. \end{aligned}$$

Then $\text{in}_{\prec}(I_X)_m^c = \Sigma_{Y,m} \cup \Sigma_{Z,m}$ and $\Sigma_{Y,m} \cap \Sigma_{Z,m} = \{x_l^m\}$. Here, $\text{in}_{\prec}(I_X)_m$ is considered as a subset of the set of all monomials of degree m in $k[x_0, \dots, x_n]$.

Proof. Let M_1 be the set of degree m monomials in $k[x_0, \dots, x_l]$ and M_2 , the set of degree m monomials in $k[x_1, \dots, x_n]$. Then by definition of T , $T^c = M_1 \cup M_2$ and $M_1 \cap M_2 = \{x_l^m\}$. By Lemma 2.1, we have

$$\text{in}_{\prec}(I_X)_m^c = \text{in}_{\prec}(I_Y \cap k[x_1, \dots, x_n])_m^c \cap \text{in}_{\prec}(I_Z \cap k[x_0, \dots, x_l])_m^c \cap (M_1 \cup M_2)$$

Since $M_1 \cap k[x_1, \dots, x_n] = \{x_l^m\}$ and $x_l^m \notin I_Y$, we have $M_1 \subset \text{in}_{\prec}(I_Y \cap k[x_1, \dots, x_n])_m^c$. Likewise, $M_2 \subset \text{in}_{\prec}(I_Z \cap k[x_0, \dots, x_l])_m^c$ and it follows that

$$(\text{in}_{\prec}(I_Y \cap k[x_1, \dots, x_n])_m^c \cap M_2) \cup (\text{in}_{\prec}(I_Z \cap k[x_0, \dots, x_l])_m^c \cap M_1) = \Sigma_{Y,m} \cup \Sigma_{Z,m}.$$

That $\Sigma_{Y,m} \cap \Sigma_{Z,m} = M_1 \cap M_2 = \{x_l^m\}$ is clear from the fact that x_l^m is the only degree m monomial not vanishing at p , and thus not contained in any of the initial ideals involved in the discussion. \square

3. STATE POLYTOPE DECOMPOSITION FORMULA

We prove the main result in this section. Surely, it suffices to prove Theorem 1.2 for the $\ell = 2$ case, as the general case would follow from it by a simple induction. So, let $X = Y \cup Z$ and T be as in Lemma 2.1. We shall prove that

$$(3) \quad \text{State}_m(I_X) = \text{State}_m(I_Y \cap k[x_1, \dots, x_n]) + \text{State}_m(I_Z \cap k[x_0, \dots, x_l]) + \gamma$$

where $\gamma = \sum_{x^\alpha \in T} \alpha$.

Let $\sum_{x^\alpha \in \text{in}_{\prec}(I_X)_m} \alpha$ be a vertex of $\text{State}_m(I_X)$ induced by some monomial order \prec on $k[x_0, \dots, x_n]$. Recall the notations

$$\Sigma_{Y,m}^c = \text{in}_{\prec}(I_Y \cap k[x_1, \dots, x_n])_m, \quad \Sigma_{Z,m}^c = \text{in}_{\prec}(I_Z \cap k[x_0, \dots, x_l])_m.$$

By Lemma 2.1, we have

$$\text{in}_{\prec}(I_X) = \text{in}_{\prec}(I_Y \cap k[x_1, \dots, x_n]) + \text{in}_{\prec}(I_Z \cap k[x_0, \dots, x_l]) + T$$

which implies

$$\sum_{\alpha \in \text{in}_{\prec} (I_X)_m} \alpha = \sum_{\alpha \in \Sigma_{Y,m}^c} \alpha + \sum_{\alpha \in \Sigma_{Z,m}^c} \alpha + \sum_{\alpha \in T} \alpha.$$

Since $\sum_{\alpha \in \Sigma_{Y,m}^c} \alpha$ and $\sum_{\alpha \in \Sigma_{Z,m}^c} \alpha$ are vertices of $\text{State}_m(I_Y \cap k[x_1, \dots, x_n])$ and $\text{State}_m(I_Z \cap k[x_0, \dots, x_l])$ respectively, $\sum_{\alpha \in \text{in}_{\prec} (I_X)_m} \alpha$ is contained in the right hand side of (3).

Conversely, let α_1 be a vertex of $\text{State}_m(I_Y \cap k[x_1, \dots, x_n])$ induced by the monomial order \prec_1 on $k[x_1, \dots, x_n]$ and α_2 , a vertex of $\text{State}_m(I_Z \cap k[x_0, \dots, x_l])$ induced by the monomial order \prec_2 on $k[x_0, \dots, x_l]$. We claim that there is a monomial order \prec on $k[x_0, \dots, x_n]$ that induces the initial ideals with respect to the given orders \prec_1 on $k[x_1, \dots, x_n]$ and \prec_2 on $k[x_0, \dots, x_l]$. There are vectors $v \in \mathbb{N}^{n-l+1}$ and $v' \in \mathbb{N}^{l+1}$ such that (cf. [Stu96, Proposition 1.11]).

$$\begin{aligned} \text{in}_v(I_Y \cap k[x_1, \dots, x_n]) &= \text{in}_{\prec_1}(I_Y \cap k[x_1, \dots, x_n]) \\ \text{in}_{v'}(I_Z \cap k[x_0, \dots, x_l]) &= \text{in}_{\prec_2}(I_Z \cap k[x_0, \dots, x_l]) \end{aligned}$$

By modifying the first entry of v (without affecting the order it defines), we may assume that it equals the last entry of v' . For instance, we may simply add $v'_l - v_1$ to all entries of v , which does not change the monomial order. Let $w = (v'_1, \dots, v'_l = v_1, v_2, \dots, v_{n-l+1})$. In general, this only defines a partial order in which case we may employ any tie-breaking device, for instance the Lex order, to define a total order: declare $M = \prod_{i=0}^n x_i^{\alpha_i} \prec M' = \prod_{i=0}^n x_i^{\beta_i}$ if

- (i) $w \cdot \alpha < w \cdot \beta$; or
- (ii) $w \cdot \alpha = w \cdot \beta$ and $M \prec' M'$

where \prec' is the chosen tie-breaking. Note that $\text{in}_{\prec}(I_Y \cap k[x_1, \dots, x_n]) = \text{in}_{\prec_1}(I_Y \cap k[x_1, \dots, x_n])$ since \prec induces the weight order given by v on $k[x_1, \dots, x_n]$. Similar statement holds for I_Z and \prec_2 . Then by Lemma 2.1, we have

$$\text{in}_{\prec}(I_X) = \text{in}_{\prec_1}(I_Y \cap k[x_1, \dots, x_n]) + \text{in}_{\prec_2}(I_Z \cap k[x_0, \dots, x_l]) + T$$

and this shows that $\alpha_1 + \alpha_2 + \gamma = \sum_{\alpha \in \text{in}_{\prec} (I_X)_m} \alpha \in \text{State}_m(I_X)$. This completes the proof of Theorem 1.2.

We now prove the decomposition formula for the Hilbert-Mumford index.

Lemma 3.1. *Let X, Y, Z be as in Lemma 2.1, $\rho : \mathbb{G}_m \rightarrow \text{GL}_{n+1}$ be a 1-parameter subgroup diagonalized by $\{x_0, \dots, x_n\}$ with weights (r_0, r_1, \dots, r_n) , and ρ', ρ'' be the restrictions of ρ to $\text{GL}(kx_1 + \dots + kx_n)$ and $\text{GL}(kx_0 + \dots + kx_l)$, respectively. Then the Hilbert-Mumford index $\mu([X]_m, \rho)$ of the m th Hilbert point of X with respect to ρ is given by*

$$\begin{aligned} \mu([X]_m, \rho) &= \mu([Y]_m, \rho') + \mu([Z]_m, \rho'') - \frac{mP_Z(m)}{n+1-l} \sum_{i=l}^n r_i - \frac{mP_Y(m)}{l+1} \sum_{i=0}^l r_i \\ &\quad + \frac{mP_X(m)}{n+1} \sum_{i=0}^n r_i + mr_l \end{aligned}$$

where

$$P_Y(m) = \dim_k(k[x_1, \dots, x_n] / \text{in}_{\prec_{\rho'}}(I_Y \cap k[x_1, \dots, x_n]))_m$$

and

$$P_Z(m) = \dim_k(k[x_0, \dots, x_l] / \text{in}_{\prec_{\rho''}}(I_Z \cap k[x_0, \dots, x_l]))_m$$

are the Hilbert polynomials of Y and Z regarded as closed subvarieties of $\{x_0 = \dots = x_{l-1} = 0\} \simeq \mathbb{P}^{n-l}$ and $\{x_{l+1} = \dots = x_n = 0\} \simeq \mathbb{P}^l$, respectively.

Proof. Let $\Sigma_{Y,m} = \text{in}_{\prec_\rho}(I_Y \cap k[x_1, \dots, x_n])_m^c$ and $\Sigma_{Z,m} = \text{in}_{\prec_{\rho''}}(I_Z \cap k[x_0, \dots, x_l])_m^c$ where complements are taken in $k[x_1, \dots, x_n]_m$ and $k[x_0, \dots, x_l]_m$, respectively. By [HHL10], the Hilbert-Mumford index can be computed by the formula

$$(4) \quad \mu([X]_m, \rho) = - \sum_{x^\alpha \notin \text{in}_{\prec_\rho}(I_X)_m} \text{wt}(x^\alpha) + \frac{mP_X(m)}{n+1} \sum_{i=0}^n r_i.$$

Since $\text{in}_{\prec_\rho}(I_X)_m^c = \Sigma_{Y,m} \cup \Sigma_{Z,m}$ and $\Sigma_{Y,m} \cap \Sigma_{Z,m} = \{x_l^m\}$, we have

$$- \sum_{x^\alpha \notin \text{in}_{\prec_\rho}(I_X)_m} \text{wt}(x^\alpha) = - \sum_{x^\alpha \in \Sigma_{Y,m}} \text{wt}(x^\alpha) - \sum_{x^\alpha \in \Sigma_{Z,m}} \text{wt}(x^\alpha) + \text{wt}(x_l^m).$$

By using [HHL10] again, we have

$$\begin{aligned} \mu([Y]_m, \rho') &= \frac{mP_Y(m)}{n-l+1} \sum_{i=l}^n r_i - \sum_{x^\alpha \in \Sigma_{Y,m}} \text{wt}(x^\alpha) \\ \mu([Z]_m, \rho'') &= \frac{mP_Z(m)}{l+1} \sum_{i=0}^l r_i - \sum_{x^\alpha \in \Sigma_{Z,m}} \text{wt}(x^\alpha) \end{aligned}$$

and plugging these in the Equation (4) produces the desired formula. \square

In terms of monomial weights, Proposition 1.8 takes the following form:

$$(5) \quad \mu([X]_m, \rho) = - \sum_{x^\alpha \in \Sigma_{Y,m}} \text{wt}_{\rho'}(x^\alpha) - \sum_{x^\alpha \in \Sigma_{Z,m}} \text{wt}_{\rho''}(x^\alpha) + \frac{mP(m)}{n+1} + mr_l.$$

Using the lemma inductively, we obtain the general case Proposition 1.8 immediately.

4. GIT OF HILBERT POINTS OF PLURICANONICAL CURVES

Our main application is to the study of GIT of pluricanonical curves.

4.1. Bi-canonical elliptic bridge. We revisit the state polytope analysis in [MS11, Example 8.4]. Morrison and Swinarski considers the state polytope of a genus five curve of the form $C = \mathcal{W}_2 \cup E \cup \mathcal{W}_2$ where \mathcal{W}_g denotes the Wiman curve of genus g (Section 6.2, *ibid*) and E is the elliptic curve $y^2 = x^3 - x$. According to the direct computation using the ideal of C , it has 500,094 initial ideals. By using the decomposition formula and Macaulay 2, we can compute its state polytope rather easily since the state polytopes of \mathcal{W}_2 and E are fairly small. Here, we give a complete description of the second state polytope.

$$\begin{array}{ccccc}
& \text{Conv} & & & \text{Conv} \\
& \underbrace{(2, 2, 1, 1, 2, \mathbf{0}_7)} & & & \underbrace{(\mathbf{0}_7, 2, 1, 1, 2, 2)} \\
& (2, 2, 0, 3, 1, \mathbf{0}_7) & & & (\mathbf{0}_7, 3, 2, 1, 2, 0) \\
& (2, 1, 1, 4, 0, \mathbf{0}_7) & & & (\mathbf{0}_7, 2, 1, 1, 4, 0) \\
& (2, 0, 3, 3, 0, \mathbf{0}_7) & & & (\mathbf{0}_7, 1, 3, 0, 4, 0) \\
& (0, 2, 3, 3, 0, \mathbf{0}_7) & & & (\mathbf{0}_7, 2, 4, 0, 2, 0) \\
& (0, 0, 3, 4, 1, \mathbf{0}_7) & & & (\mathbf{0}_7, 1, 5, 1, 1, 0) \\
& (0, 3, 1, 4, 0, \mathbf{0}_7) & \text{Conv} & & (\mathbf{0}_7, 0, 4, 1, 3, 0) \\
& (0, 1, 1, 5, 1, \mathbf{0}_7) & \underbrace{(\mathbf{0}_4, 2, 0, 1, 1, \mathbf{0}_4)} & & (\mathbf{0}_7, 1, 3, 0, 2, 2) \\
& (0, 4, 0, 3, 1, \mathbf{0}_7) & (\mathbf{0}_4, 1, 0, 3, 0, \mathbf{0}_4) & & (\mathbf{0}_7, 0, 4, 1, 1, 2) \\
& (0, 2, 0, 4, 2, \mathbf{0}_7) & (\mathbf{0}_4, 0, 0, 3, 1, \mathbf{0}_4) & + & (\mathbf{0}_7, 2, 0, 3, 3, 0) \\
& (2, 1, 3, 0, 2, \mathbf{0}_7) & (\mathbf{0}_4, 1, 0, 1, 2, \mathbf{0}_4) & & (\mathbf{0}_7, 1, 1, 4, 2, 0) \\
& (2, 0, 4, 1, 1, \mathbf{0}_7) & (\mathbf{0}_4, 1, 2, 0, 1, \mathbf{0}_4) & & (\mathbf{0}_7, 3, 1, 3, 1, 0) \\
& (0, 2, 4, 1, 1, \mathbf{0}_7) & \underbrace{(\mathbf{0}_4, 0, 2, 2, 0, \mathbf{0}_4)} & & (\mathbf{0}_7, 2, 0, 3, 1, 2) \\
& (0, 0, 4, 2, 2, \mathbf{0}_7) & & & (\mathbf{0}_7, 1, 1, 4, 0, 2) \\
& (0, 3, 3, 0, 2, \mathbf{0}_7) & & & (\mathbf{0}_7, 2, 2, 4, 0, 0) \\
& (0, 1, 3, 1, 3, \mathbf{0}_7) & & & (\mathbf{0}_7, 1, 4, 3, 0, 0) \\
& (0, 4, 1, 1, 2, \mathbf{0}_7) & & & (\mathbf{0}_7, 0, 3, 3, 2, 0) \\
& \underbrace{(0, 2, 1, 2, 3, \mathbf{0}_7)} & & & \underbrace{(\mathbf{0}_7, 0, 3, 3, 0, 2)}
\end{array}$$

The 2nd state polytope of C is the $\gamma = (7, 7, 7, 7, 4, 8, 8, 4, 7, 7, 7, 7)$ -translate of Minkowski sum as given above, where $\mathbf{0}_i = (\underbrace{0, \dots, 0}_i)$. The three columns are the

(2nd) state polytopes of the three components \mathcal{W}_2 , E and \mathcal{W}_2 . By Corollary 1.4, the vertices of the Minkowski sum are precisely the sums of three vertices obtained by choosing one from each column. Hence the 2nd state polytope of C has $18 \cdot 18 \cdot 6 = 1944$ vertices. We note that Minkowski sums can be effectively computed by `polymake`, verifying our formal computation above.

4.2. 4-canonical curve with a cuspidal tail. We revisit the computation in [HM10, Lemma 3] where the instability of a 4-canonically embedded curve

$$Y = D \cup_{\mathbb{P}} R \hookrightarrow \mathbb{P}(\Gamma(\omega_Y^{\otimes 4})) \simeq \mathbb{P}^n$$

of genus $g \geq 2$ with a rational cuspidal tail R is proved. Here, $n = 7(g - 1)$ and the isomorphism is given by choosing sections (homogeneous coordinates) such that

$$\begin{aligned}
R &\subset \{x_0 = \dots = x_{7g-11} = 0\} \\
D &\subset \{x_{7g-9} = \dots = x_n = 0\}.
\end{aligned}$$

Let $l = 7g - 10$ and apply Proposition 1.8. Let ρ be the one-parameter subgroup with weight $(4, 4, \dots, 4, 3, 2, 0)$. The total ρ -weight of the degree two monomials in x_1, \dots, x_n that are not in $\text{in}_{\rho}(R)$ is 35. The degree two monomials in x_0, \dots, x_l not in $\text{in}_{\rho}(D)$ contribute $15g - 22$ to the total weight. Lastly, $mP(m)/(n + 1) = m(8m - 1)(4g - 5)$. Putting all these together through Proposition 1.8 or the monomial weight version (5), we get

$$\mu([Y]_2, \rho) = -(35 + 2 \cdot 4 \cdot (15g - 22)) + 30 \cdot (4g - 5) + 2 \cdot 4 = -1.$$

Likewise,

$$\mu([Y]_3, \rho) = -(77 + 3 \cdot 4 \cdot (23g - 34)) + 3 \cdot 23 \cdot (4g - 5) + 3 \cdot 4 = -2$$

from which it is deduced that $\mu([Y]_m, \rho) = -m + 1$.

4.3. Open rosaries. We revisit the Hilbert-Mumford index computation of the special curves called ‘rosaries’. Recall from [HH13, Definition 6.1] that an open rosary of genus r is $R = L_1 \cup_{a_1} L_2 \cup_{a_2} \dots \cup_{a_r} L_{r+1}$ where L_i are smooth genus zero curves and a_i ’s are tacnodes. Each L_i intersects with L_j if and only if $|i - j| = 1$. In [HH13, Section 8.1], the Hilbert-Mumford index of an open rosary (with respect to a 1-ps coming from its automorphism group) embedded by $\omega_R^{\otimes 2}(2a_0 + 2a_{r+1})$ is computed, where a_0 and a_{r+1} are smooth points of L_1 and L_{r+1} , respectively. Note that since a_1, \dots, a_r are tacnodes, we are not able to apply the decomposition formula directly. However, in the proposition below, we shall demonstrate that a similar argument can be used to obtain a systematic analysis of the initial ideal.

Proposition 4.1. *The initial ideal of R with respect to the ρ -weighted Lex order satisfies the following decomposition: Let T_l^d denote the set of degree d monomials in x_0, \dots, x_{3l+2} (resp. x_0, \dots, x_{3r}) which involve x_i and x_j for $i < 3l - 2$ and $j > 3l - 1$ for $l = 1, \dots, r - 1$ (resp. $l = r$). We define $x_l = x_0$ if $l < 0$ and $x_l = x_{3r}$ if $l > 3r$.*

(i) (degree 2 piece)

$$(\text{in } I_R)_2 \cup \{x_{3l-2}^2\}_{l=1}^r = \cup_{l=1}^{r+1} \text{in } (I_{L_l} \cap k[x_{3l-5}, \dots, x_{3l-1}])_2 \cup (\cup_{l=1}^r T_l^2)$$

(ii) (degree 3 piece)

$$(\text{in } I_R)_3 \cup \{x_{3l-2}^3, x_{3l-2}^2 x_{3l-1}\}_{l=1}^r = \cup_{l=1}^{r+1} \text{in } (I_{L_l} \cap k[x_{3l-5}, \dots, x_{3l-1}])_3 \cup (\cup_{l=1}^r T_l^3)$$

Proof. The proof follows the idea of Lemma 2.1 closely, but more care needs to be exercised because of the additional overlapping coordinates.

(i) (degree 2 piece) Suppose that $x^\alpha \in (\text{in } I_R)_2$, then $x^\alpha = \text{in } f$ for some homogeneous quadratic $f \in I_R$. If x^α is not contained in $\cup_{l=1}^r T_l$, then x^α is contained in $k[x_{3l-5}, \dots, x_{3l-1}]$ for some l . Let $g = f(0, \dots, 0, x_{3l-5}, \dots, x_{3l-1}, 0, \dots, 0)$. Then $g \in I_{L_l} \cap k[x_{3l-5}, \dots, x_{3l-1}]$ and therefore, x^α is contained in $(\text{in } I_{L_l} \cap k[x_{3l-5}, \dots, x_{3l-1}])_2$. Since $x_{3l-2}^2 - x_{3l-1}x_{3l} \in I_{L_{l+1}}$, we have $x_{3l-2}^2 \in \text{in } I_{L_{l+1}}$. For the other inclusion, suppose that $x^\alpha \in (\text{in } I_{L_l} \cap k[x_{3l-5}, \dots, x_{3l-1}])_2$ for some $l = 1, 2, \dots, r + 1$. If $l = 1$ or $r + 1$, then $x^\alpha = x_0x_2$ or x_{3r-2}^2 respectively, but we know that $x_0x_2 = \text{in } (x_0x_2 - x_1^2 - x_2x_3) \in \text{in } I_R$. Hence x^α is contained in the left hand side.

If $l = 2, 3, \dots, r$, then

$$I_{L_l} = \langle x_{3l-2}^2 - x_{3l-3}x_{3l-1}, x_{3l-3}x_{3l-2} - x_{3l-5}x_{3l-1}, x_{3l-5}x_{3l-2} - x_{3l-4}x_{3l-1}, x_{3l-3}^2 - x_{3l-4}x_{3l-1}, x_{3l-5}x_{3l-3} - x_{3l-4}x_{3l-2}, x_{3l-5}^2 - x_{3l-4}x_{3l-3} \rangle.$$

Let’s denote the generators of I_{L_l} by f_i for $i = 1, 2, \dots, 6$ in the order shown above. The initial ideal of L_l is

$$\text{in } I_{L_l} = \langle x_{3l-3}x_{3l-1}, x_{3l-4}x_{3l-1}, x_{3l-4}x_{3l-2}^2, x_{3l-5}x_{3l-1}, x_{3l-5}x_{3l-2}, x_{3l-5}x_{3l-3}, x_{3l-5}^2 \rangle$$

This shows that x^α comes from the initial term of some generator of I_{L_l} . That is, $x^\alpha = \text{in } (f_i)$ for some i . But $f_i \in I_R$ for $i = 2, 3, 4, 5$, it suffices to consider the case $i = 1, 6$. For the case $i = 1$, then $x^\alpha = x_{3l-3}x_{3l-1}$ and so $x^\alpha \in \text{in } (x_{3l-2}^2 - x_{3l-3}x_{3l-1} - x_{3l-1}x_{3l}) \in \text{in } I_R$. For $i = 6$, then $x^\alpha = x_{3l-5}^2$. Obviously, $\cup_{l=1}^r T_l$ is contained in $\text{in } I_R$.

(ii) (degree 3 piece) Since $x_{3l-2}^2 \in \text{in } I_{L_{l+1}}$, x_{3l-2}^3 and $x_{3l-2}^2 x_{3l-1}$ are also elements in $I_{L_{l+1}}$. If $x^\alpha \in \text{in } I_R$ and not in $\cup_{l=1}^r T_l$, we can easily show that x^α is contained in the right hand side by the same proof of the degree 2 case.

For the other inclusion, suppose that x^α is an element in $(I_{L_l} \cap k[x_{3l-5}, \dots, x_{3l-1}])_3$ for some l . If $l = 1$, then $x^\alpha = x_i(x_0 x_2)$ for some $i = 0, 1, 2$. Hence $x^\alpha = \text{in}(x_i(x_0 x_2 - x_1^2 - x_2 x_3)) \in \text{in } I_R$. If $l = r + 1$, then $x^\alpha = x_i(x_{3r-2})^2$ for some $i = 3r - 2, 3r - 1, 3r$. For $i = 3r$, we have $x^\alpha = \text{in}(x_{3r}(x_{3r-2}^2 - x_{3r-1} x_{3r})) \in \text{in } I_R$. Suppose that $l = 2, 3, \dots, r$, then by using the same notation in degree 2 case, we know that $x^\alpha = \text{in}(x_i f_j)$ for $j = 1, 2, \dots, 6$ and $i = 3l - 5, \dots, 3l - 1$ except for the case $x^\alpha = x_{3l-4} x_{3l-2}^2$. If $j = 2, 3, 4, 5$, then $x_i f_j \in I_R$ and so $x^\alpha \in \text{in } I_R$. If $j = 1$, then $x^\alpha = \text{in}(x_i(x_{3l-2}^2 - x_{3l-3} x_{3l-1} - x_{3l-1} x_{3l})) \in \text{in } I_R$. Hence it remains to consider the case $j = 6$ and $x^\alpha = x_{3l-4} x_{3l-2}^2$.

For the first case, suppose that $x^\alpha = \text{in}(x_i f_6) = x_i x_{3l-5}^2$. For $i = 3l - 5, 3l - 4$, there is nothing to prove. If $i > 3l - 4$, then $x^\alpha = \text{in}(x_{3l-5} f_j)$ for $j = 2, 3, 5$. Hence $x^\alpha \in \text{in } I_R$ since $f_j \in I_R$ where $j = 2, 3, 5$.

For the second, $x^\alpha = x_{3l-4} x_{3l-2}^2 = \text{in}\{x_{3l-4}(f_1 - x_{3l-1} x_{3l}) + x_{3l-3}(f_4)\} \in \text{in } I_R$ since $f_1 - x_{3l-1} x_{3l}$ and f_4 are elements in I_R . Finally, $\cup_{l=1}^r T_l$ is contained in $\text{in } I_R$. This completes the proof. \square

By using the above proposition, we can compute inductively the sum of ρ -weights of degree 2 and 3 monomials in the initial ideal of I_R . Let $w_i(r)$ be the sum $\sum_{x^\alpha \in \text{in}(I_R)_i} \text{wt}_\rho(x^\alpha)$ for $i = 2, 3$ where R is the open rosary of length $r + 1$. Then

$$w_2(r) = \begin{cases} w_2(r-2) + (72r - 110) & \text{for odd } r > 1 \\ w_2(r-2) + (72r - 92) & \text{for even } r > 2 \end{cases}$$

For the degree 3 case,

$$w_3(r) = \begin{cases} w_3(r-2) + (162r^2 - 162r - 57) & \text{for odd } r > 1 \\ w_3(r-2) + (162r^2 - 108r - 66) & \text{for even } r > 2 \end{cases}$$

Since $w_2(1) = 6, w_2(2) = 52, w_3(1) = 34, w_3(2) = 366$, we can easily compute the $w_i(r)$ as follows.

$$w_2(r) = \begin{cases} 18r^2 - 19r + 7 & \text{for odd } r \\ 18r^2 - 10r & \text{for even } r \end{cases}$$

and

$$w_3(r) = \begin{cases} 27r^3 + \frac{81}{2}r^2 - \frac{111}{2}r + 22 & \text{for odd } r \\ 27r^3 + 54r^2 - 33r & \text{for even } r \end{cases}$$

which is consistent with the results in [HH13, Section 8.1].

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